Expected number of components of random polynomial lemniscates

Subhajit Ghosh

Postdoctoral Researcher
Department of Mathematics, Bar-Ilan University
Ramat Gan, Tel Aviv District, Israel, 52900

Probability Seminar

Einstein Institute of Mathematics, The Hebrew University of Jerusalem.

May 19, 2026

Introduction To Polynomial Lemniscate:

Let $p(z)$ be a monic complex polynomial. The R-*lemniscate* of p is defined by

$$\Lambda_p(R) := \{z \in \mathbb{C} : |p(z)| < R\}, \qquad \Lambda_p := \{z \in \mathbb{C} : |p(z)| < 1\}.$$

R-Lemniscate (Unit) Lemniscate

Introduction To Polynomial Lemniscate:

Let $p(z)$ be a monic complex polynomial. The R-*lemniscate* of p is defined by

$$\Lambda_p(R) := \{z \in \mathbb{C} : |p(z)| < R\}, \qquad \Lambda_p := \{z \in \mathbb{C} : |p(z)| < 1\}.$$

R-Lemniscate (Unit) Lemniscate

Basic properties:

- Λ_p is a bounded open set with rectifiable boundary.

Introduction To Polynomial Lemniscate:

Let $p(z)$ be a monic complex polynomial. The R-*lemniscate* of p is defined by

$$\Lambda_p(R) := \{z \in \mathbb{C} : |p(z)| < R\}, \qquad \Lambda_p := \{z \in \mathbb{C} : |p(z)| < 1\}.$$

R-Lemniscate　　(Unit) Lemniscate

Basic properties:

- Λ_p is a bounded open set with rectifiable boundary.

$$Z_p \subset \Lambda_p$$

zero set of p　　lemniscate of p

Introduction To Polynomial Lemniscate:

Let $p(z)$ be a monic complex polynomial. The R-*lemniscate* of p is defined by

$$\Lambda_p(R) := \{z \in \mathbb{C} : |p(z)| < R\}, \qquad \Lambda_p := \{z \in \mathbb{C} : |p(z)| < 1\}.$$

R-Lemniscate (Unit) Lemniscate

Basic properties:

- Λ_p is a bounded open set with rectifiable boundary.

$$Z_p \subset \Lambda_p$$

zero set of p lemniscate of p

- Each connected component of Λ_p Contains a root of the polynomial p.

Introduction To Polynomial Lemniscate:

Let $p(z)$ be a monic complex polynomial. The R-*lemniscate* of p is defined by

$$\Lambda_p(R) := \{z \in \mathbb{C} : |p(z)| < R\}, \qquad \Lambda_p := \{z \in \mathbb{C} : |p(z)| < 1\}.$$

R-Lemniscate (Unit) Lemniscate

Basic properties:

- Λ_p is a bounded open set with rectifiable boundary.

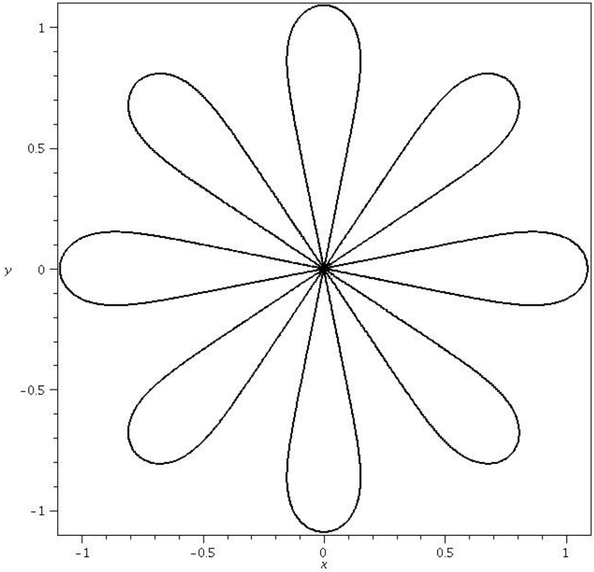
$$Z_p \subset \Lambda_p$$

zero set of p lemniscate of p

- Each connected component of Λ_p Contains a root of the polynomial p.
- Let $C(\Lambda_p) :=$ Number of components of Λ_p, then

$$1 \leq C(\Lambda_p) \leq n$$

Erdös Lemniscate: $z^n - 1$



Two Facets of Random Polynomials

A monic complex polynomial $p(z)$ of degree n can be expressed in two equivalent fundamental forms.

1. Expanded Form

$$p(z) = \sum_{j=0}^{n} c_j z^j$$

Specified by its **coefficients** $\{c_j\}_{j=0}^{n}$.

Random Coefficients Model

• Set the coefficients c_j as random variables (e.g., Kac, Kostlan, Weyl).

2. Factorised Form

$$p(z) = \prod_{j=1}^{n} (z - z_j)$$

Specified by its **zeros** $\{z_j\}_{j=1}^{n}$.

Random Roots Model

• Set the zeros z_j to be random variables.

History:(Lundberg & Ramachandran) JLMS (2017)

For the **Kac ensemble** $p_n(z) = \sum_{k=0}^{n} a_k z^k$ (i.i.d. $a_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$), Lundberg and Ramachandran (2017) established:

- **Expected Number of Components:**

$$\mathbb{E}[C(\Lambda_{p_n})] \sim n$$

History:(Lundberg & Ramachandran) JLMS (2017)

For the **Kac ensemble** $p_n(z) = \sum_{k=0}^{n} a_k z^k$ (i.i.d. $a_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$), Lundberg and Ramachandran (2017) established:

- **Expected Number of Components:**

$$\mathbb{E}[C(\Lambda_{p_n})] \sim n$$

  - The component count reach the **maximal order of growth** allowed by deterministic bounds.

History:(Lundberg & Ramachandran) JLMS (2017)

For the **Kac ensemble** $p_n(z) = \sum_{k=0}^{n} a_k z^k$ (i.i.d. $a_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$), Lundberg and Ramachandran (2017) established:

- **Expected Number of Components:**

$$\mathbb{E}[C(\Lambda_{p_n})] \sim n$$

  - The component count reach the **maximal order of growth** allowed by deterministic bounds.
  - Since the zeros of Kac polynomials converge to unit circle, the components concentrate near $\{|z| = 1\}$.

History:(Lundberg & Ramachandran) JLMS (2017)

For the **Kac ensemble** $p_n(z) = \sum_{k=0}^{n} a_k z^k$ (i.i.d. $a_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$), Lundberg and Ramachandran (2017) established:

- **Expected Number of Components:**

$$\mathbb{E}[C(\Lambda_{p_n})] \sim n$$

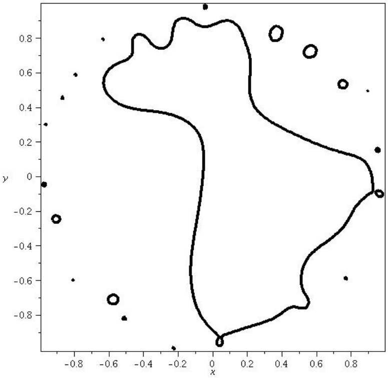
  - The component count reach the **maximal order of growth** allowed by deterministic bounds.
  - Since the zeros of Kac polynomials converge to unit circle, the components concentrate near $\{|z| = 1\}$.

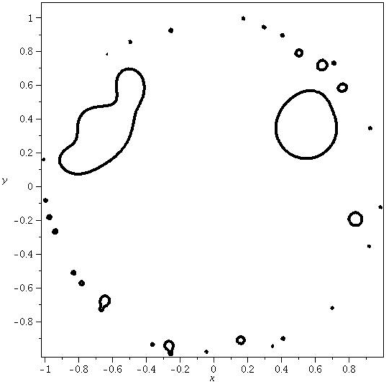
- **Existence of Giant Component:**
  - For any fixed $r \in (0,1)$, there is a **positive probability** (independent of n) that the lemniscate has a component that contains a ball of radius r.

$$\mathbb{P}\left(B(0,r) \subset \Lambda_{p_n}\right) > c(r) > 0$$

Probability of having a gaint component

Lemniscate of Kac the Polynomials





Random Setting

- Let $\{X_i(\omega)\}_{i=1}^{\infty}$ be i.i.d. random variables with law μ. We define

$$P_n(z) := P_n(z, \omega) := \prod_{i=1}^{n} \big(z - X_i(\omega)\big)$$

random polynomial

Random Setting

- Let $\{X_i(\omega)\}_{i=1}^{\infty}$ be i.i.d. random variables with law μ. We define

$$P_n(z) := P_n(z, \omega) := \prod_{i=1}^{n} \big(z - X_i(\omega)\big)$$

random polynomial

- The associated random lemniscate is

$$\Lambda_n := \Lambda_n(\omega) := \{z \in \mathbb{C} : |P_n(z, \omega)| < 1\}$$

random lemniscate

Random Setting

- Let $\{X_i(\omega)\}_{i=1}^{\infty}$ be i.i.d. random variables with law μ. We define

$$P_n(z) := P_n(z,\omega) := \prod_{i=1}^{n}\big(z - X_i(\omega)\big)$$

random polynomial

- The associated random lemniscate is

$$\Lambda_n := \Lambda_n(\omega) := \{z \in \mathbb{C} : |P_n(z,\omega)| < 1\}$$

random lemniscate

- The number of connected components is the random variable

$$C(\Lambda_n) := C(\Lambda_n)(\omega)$$

number of components

Random Setting

- Let $\{X_i(\omega)\}_{i=1}^{\infty}$ be i.i.d. random variables with law μ. We define

$$P_n(z) := P_n(z, \omega) := \prod_{i=1}^{n} \big(z - X_i(\omega)\big)$$

random polynomial

- The associated random lemniscate is

$$\Lambda_n := \Lambda_n(\omega) := \{z \in \mathbb{C} : |P_n(z, \omega)| < 1\}$$

random lemniscate

- The number of connected components is the random variable

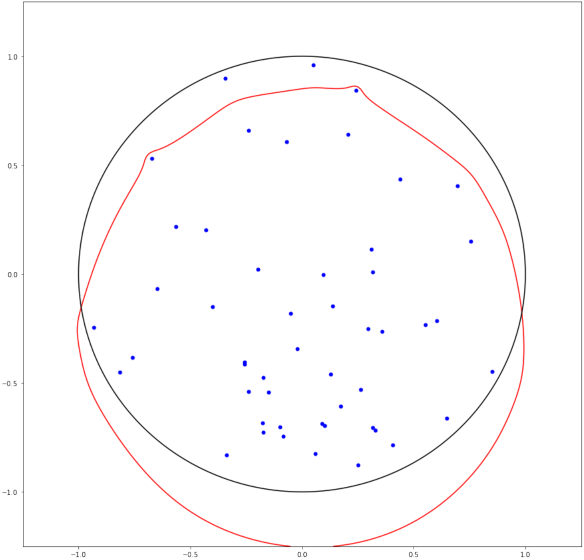
$$C(\Lambda_n) := C(\Lambda_n)(\omega)$$

number of components

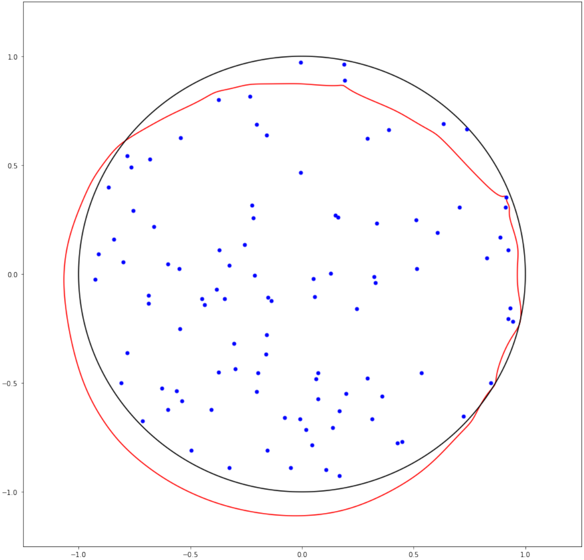
- **Goal:** To investigate the asymptotic behaviour of $\mathbb{E}[C(\Lambda_n)]$ for

(i) $\mu = \mathrm{unif}(\mathbb{D})$ & (ii) $\mu = \mathrm{unif}(\mathbb{S}^1)$

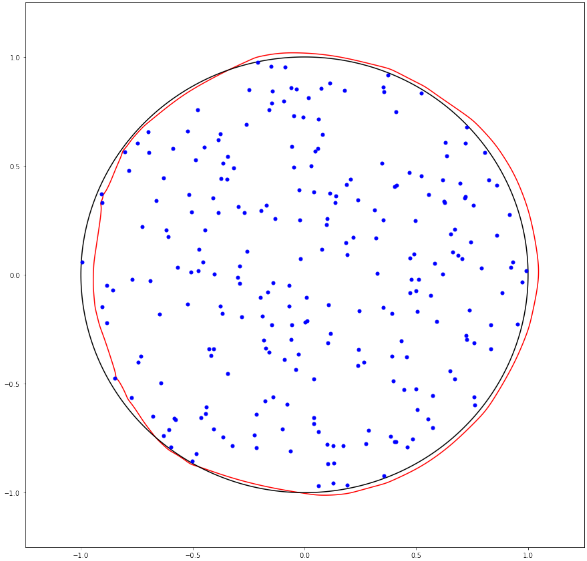
Simulations: $n = 50$, $\mu = \mathrm{unif}(\mathbb{D})$



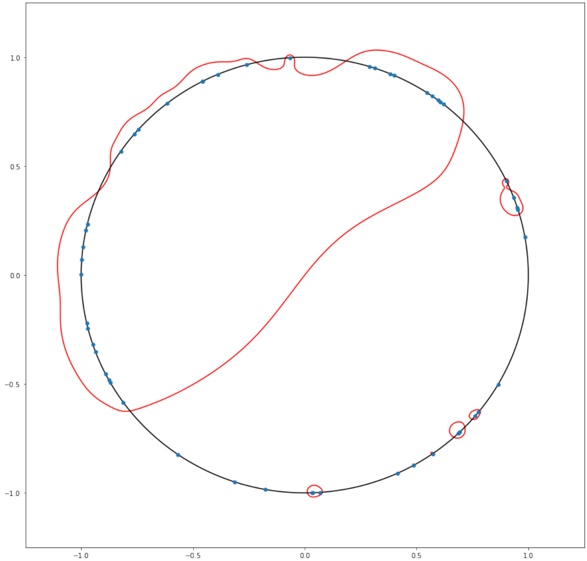
Simulations: $n = 100$, $\mu = \mathrm{unif}(\mathbb{D})$



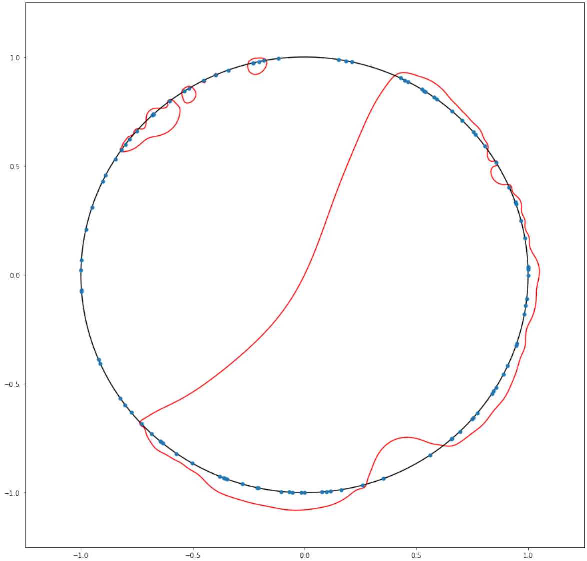
Simulations: $n = 250$, $\mu = \mathrm{unif}(\mathbb{D})$



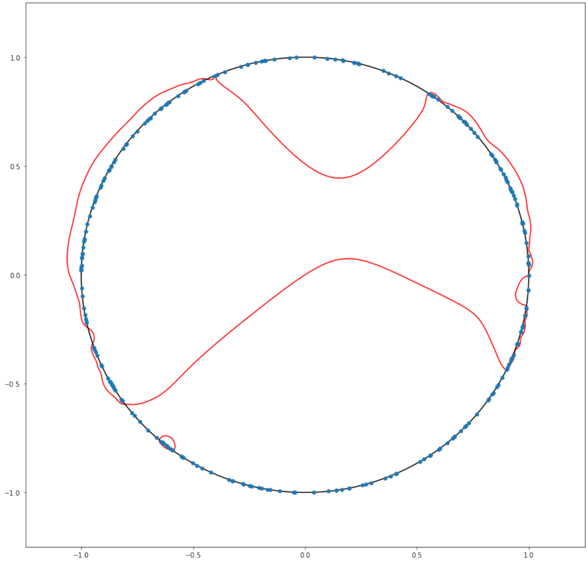
Simulations: $n = 50$, $\mu = \mathrm{unif}(\mathbb{S}^1)$



Simulations: $n = 100$, $\mu = \mathrm{unif}(\mathbb{S}^1)$



Simulations: $n = 250$, $\mu = \mathrm{unif}(\mathbb{S}^1)$



Main Results:

Theorem 3 (S. Ghosh, 2024, EJP)

Let $\mu = \mathrm{unif}(\mathbb{D})$. Then there exist constants $C_1, C_2 > 0$ such that

$$C_1\sqrt{n} \leq \mathbb{E}[C(\Lambda_n)] \leq C_2\sqrt{n}$$

Main Results:

Theorem 3 (S. Ghosh, 2024, EJP)

Let $\mu = \mathrm{unif}(\mathbb{D})$. Then there exist constants $C_1, C_2 > 0$ such that

$$C_1\sqrt{n} \leq \mathbb{E}[C(\Lambda_n)] \leq C_2\sqrt{n}$$

Theorem 4 (S. Ghosh, 2024, EJP)

Let $\mu = \mathrm{unif}(\mathbb{S}^1)$. Then

$$\lim_{n\to\infty} \frac{\mathbb{E}[C(\Lambda_n)]}{n} = \frac{1}{2}$$

Potential Theory Preliminaries

- Let $K \subset \mathbb{C}$ be a non-empty compact set, and let μ be a Borel probability measure on K. Then

$$U_\mu(z) := \int_K \log|z-w|\,d\mu(w) \qquad z \in \mathbb{C}.$$

logarithmic potential

Potential Theory Preliminaries

- Let $K \subset \mathbb{C}$ be a non-empty compact set, and let μ be a Borel probability measure on K. Then

$$U_\mu(z) := \int_K \log|z-w|\, d\mu(w) \qquad z \in \mathbb{C}.$$

logarithmic potential

- **Example.** If $\mu = \delta_{z_0}$, then

$$U_\mu(z) = \log|z - z_0| \qquad z \in \mathbb{C}.$$

Potential Theory Preliminaries

- Let $K \subset \mathbb{C}$ be a non-empty compact set, and let μ be a Borel probability measure on K. Then

$$U_\mu(z) := \int_K \log|z - w| \, d\mu(w) \qquad z \in \mathbb{C}.$$

logarithmic potential

- **Example.** If $\mu = \delta_{z_0}$, then

$$U_\mu(z) = \log|z - z_0| \qquad z \in \mathbb{C}.$$

- For a polynomial p $\quad \mu_p := \frac{1}{\deg(p)} \sum_{z_0 \in Z_p} \delta_{z_0}$

empirical measure

Potential Theory Preliminaries

- Let $K \subset \mathbb{C}$ be a non-empty compact set, and let μ be a Borel probability measure on K. Then

$$U_\mu(z) := \int_K \log|z-w|\, d\mu(w) \qquad z \in \mathbb{C}.$$

logarithmic potential

- **Example.** If $\mu = \delta_{z_0}$, then

$$U_\mu(z) = \log|z - z_0| \qquad z \in \mathbb{C}.$$

- For a polynomial p $\quad \mu_p := \frac{1}{\deg(p)} \sum_{z_0 \in Z_p} \delta_{z_0}$

empirical measure

$$U_{\mu_p}(z) = \frac{1}{\deg(p)} \sum_{z_0 \in Z_p} \log|z - z_0| = \frac{1}{\deg(p)} \log|p(z)|$$

potential of empirical measure

Lemniscates and Potentials

Therefore,

$$\Lambda_p = \{z \in \mathbb{C} : \log|p(z)| < 0\} = \{z \in \mathbb{C} : U_{\mu_p}(z) < 0\}.$$

Most Important Identity

Lemniscates and Potentials

Therefore,

$$\Lambda_p = \{z \in \mathbb{C} : \log|p(z)| < 0\} = \{z \in \mathbb{C} : U_{\mu_p}(z) < 0\}.$$

Most Important Identity

- If the roots of the random polynomial are chosen independently from μ, then for large degree random polynomials

$$\mu_p \approx \mu \qquad \text{and hence} \qquad U_{\mu_p} \approx U_\mu.$$

Lemniscates and Potentials

Therefore,

$$\Lambda_p = \{z \in \mathbb{C} : \log|p(z)| < 0\} = \{z \in \mathbb{C} : U_{\mu_p}(z) < 0\}.$$

Most Important Identity

- If the roots of the random polynomial are chosen independently from μ, then for large degree random polynomials

$$\mu_p \approx \mu \qquad \text{and hence} \qquad U_{\mu_p} \approx U_\mu.$$

- The probabilistic meaning: for $z \in \mathbb{C}$

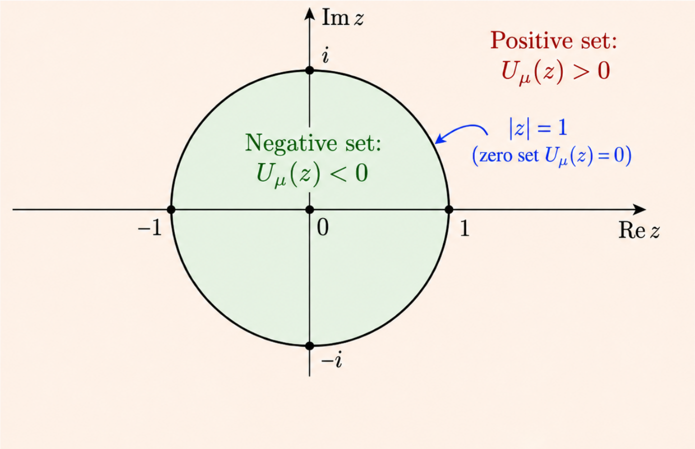
$$\mathbb{E}\left[\frac{1}{n}\log|P_n(z)|\right] = \mathbb{E}[\log|z - X_1|] = U_\mu(z), \qquad z \in \mathbb{C}.$$

Thus, heuristically,

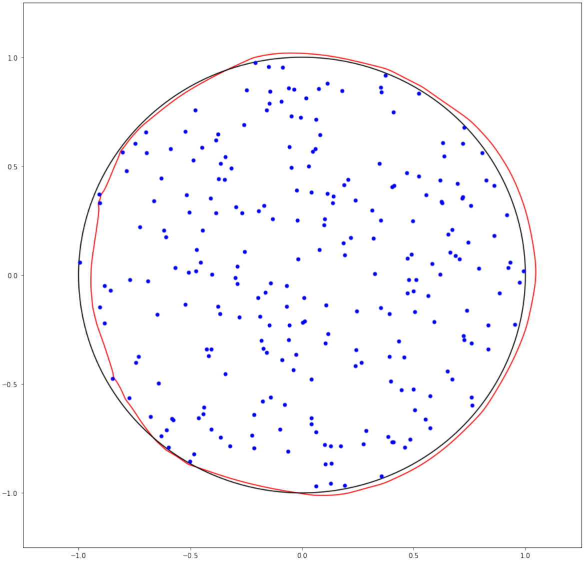
$$|P_n(z)| \approx \exp\Big(nU_\mu(z)\Big).$$

Potential function of $\mu = \mathrm{Unif}(\mathbb{D})$

$$U_\mu(z) = \begin{cases} \dfrac{|z|^2 - 1}{2}, & |z| \leqslant 1, \\ \log|z|, & |z| > 1. \end{cases}$$

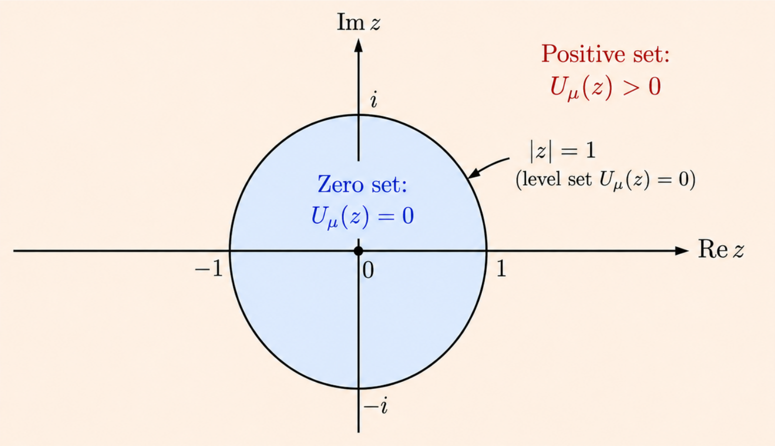


Simulations: $n = 50$, $\mu = \mathrm{unif}(\mathbb{D})$

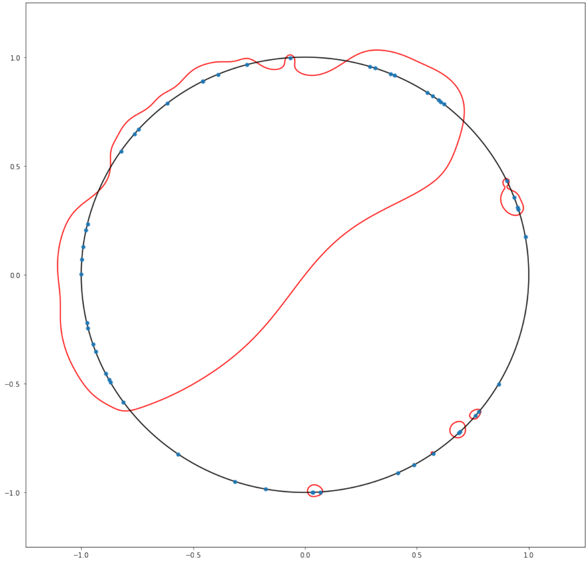


Potential function of $\mu = \mathrm{Unif}(\mathbb{S}^1)$

$$U_\mu(z) = \log^+|z| = \begin{cases} 0, & |z| \le 1, \\ \log|z|, & |z| > 1. \end{cases}$$



Simulations: $n = 50$, $\mu = \mathrm{unif}(\mathbb{S}^1)$

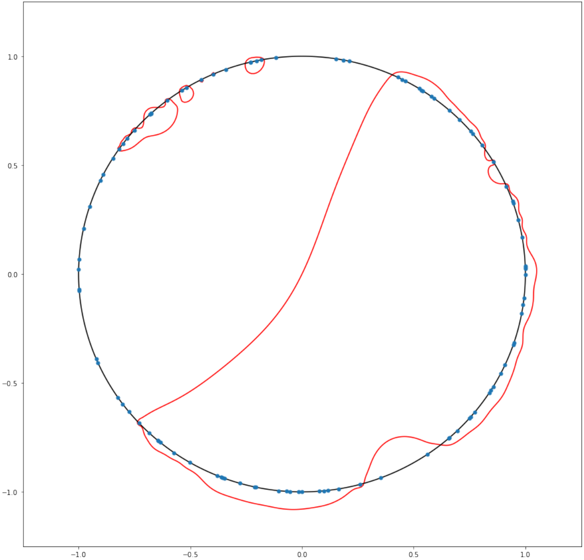


Disk Model

Proof of Lower Bound

$$C_1\sqrt{n} \leq \mathbb{E}[C(\Lambda_n)]$$

Simulations: $n = 100$, $\mu = \mathrm{unif}(\mathbb{S}^1)$



Lonely Component

- We say that a root X_j forms a *lonely component* if there exists a ball $\mathcal{B}$ containing X_j such that

$$\begin{cases} z_k \notin \mathcal{B}, \quad \forall k \neq j, \\ \\ |P_n(z)| \geq 1, \quad \forall z \in \partial\mathcal{B} \end{cases}$$

Lonely Component

- We say that a root X_j forms a *lonely component* if there exists a ball $\mathcal{B}$ containing X_j such that

$$\begin{cases} z_k \notin \mathcal{B}, & \forall k \neq j, \\ |P_n(z)| \geq 1, & \forall z \in \partial\mathcal{B} \end{cases}$$

- Define $L_i := \{X_i \text{ forms a lonely component}\}$. Then

$$C(\Lambda_n) \geq \sum_{i=1}^{n} \mathbb{1}_{L_i}.$$

Lonely Component

- We say that a root X_j forms a *lonely component* if there exists a ball $\mathcal{B}$ containing X_j such that

$$\begin{cases} z_k \notin \mathcal{B}, \quad \forall k \neq j, \\ \\ |P_n(z)| \geq 1, \quad \forall z \in \partial\mathcal{B} \end{cases}$$

- Define $L_i := \{X_i \text{ forms a lonely component}\}$. Then

$$C(\Lambda_n) \geq \sum_{i=1}^n \mathbb{1}_{L_i}.$$

- Taking expectations, we get

$$\mathbb{E}[C(\Lambda_n)] \geq \sum_{i=1}^n \mathbb{P}(L_i) = n\,\mathbb{P}(L_1)$$

Proof of Sufficient Condition

- We need to choose r small enough, such that with high probability

$$\min_{2\leq j\leq n} |X_1 - X_j| > 3r$$

Proof of Sufficient Condition

- We need to choose r small enough, such that with high probability

$$\min_{2\leq j\leq n}|X_1 - X_j| > 3r$$

- For $z \in \partial B(X_1, r)$, Taylor expansion around X_1 gives

$$|P_n(z)| = \left| P_n(X_1) + P_n'(X_1)(z - X_1) + \sum_{k=2}^{n} P_n^{(k)}(X_1)\frac{(z - X_1)^k}{k!} \right|$$

Proof of Sufficient Condition

- We need to choose r small enough, such that with high probability

$$\min_{2\leq j\leq n} |X_1 - X_j| > 3r$$

- For $z \in \partial B(X_1, r)$, Taylor expansion around X_1 gives

$$|P_n(z)| = \left| P_n(X_1) + P_n'(X_1)(z - X_1) + \sum_{k=2}^{n} P_n^{(k)}(X_1)\frac{(z-X_1)^k}{k!} \right|$$

$$\geq \left| P_n'(X_1)r \right| - \sum_{k=2}^{n} \left| P_n^{(k)}(X_1)\frac{r^k}{k!} \right|$$

Proof of Sufficient Condition

- We need to choose r small enough, such that with high probability

$$\min_{2\le j\le n}|X_1 - X_j| > 3r$$

- For $z \in \partial B(X_1, r)$, Taylor expansion around X_1 gives

$$|P_n(z)| = \left| P_n(X_1) + P_n'(X_1)(z - X_1) + \sum_{k=2}^{n} P_n^{(k)}(X_1)\frac{(z - X_1)^k}{k!} \right|$$

$$\ge \left|P_n'(X_1)r\right| - \sum_{k=2}^{n} \left| P_n^{(k)}(X_1)\frac{r^k}{k!} \right|$$

$$= |P_n'(X_1)r| \quad \left(1 - \sum_{k=2}^{n} \frac{\left|P_n^{(k)}(X_1)r^{k-1}\right|}{|P_n'(X_1)k!|}\right)$$

Sufficient Condition for a Lonely Component

- Suppose that, for the root X_1, there exists $r > 0$ (for us $r = n^{-10}$ is good enough) such that

$$\begin{cases} |P_n'(X_1)r| \gg 1 & \text{dominant linear term} \\ \left|\frac{P_n^{(k)}(X_1)r^{k-1}}{P_n'(X_1)k!}\right| < \frac{1}{2n^2}, \quad k = 2, \dots, n & \text{higher-order terms are small} \end{cases}$$

Sufficient Condition for a Lonely Component

- Suppose that, for the root X_1, there exists $r > 0$ (for us $r = n^{-10}$ is good enough) such that

$$\begin{cases} |P_n'(X_1) r| \gg 1 & \text{dominant linear term} \\ \left| \frac{P_n^{(k)}(X_1) r^{k-1}}{P_n'(X_1) k!} \right| < \frac{1}{2n^2}, \qquad k = 2, \ldots, n & \text{higher-order terms are small} \end{cases}$$

- Notice that, conditioned on $X_1 = z$ and, for $k = 2, \ldots, n$,

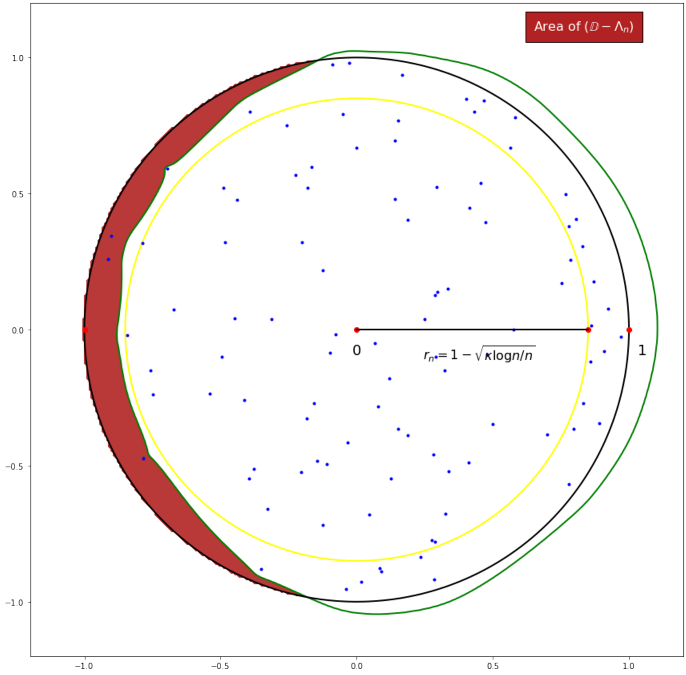
$$1) \quad |P_n'(X_1)| = \prod_{j=2}^{n} |z - X_j|$$

$$2) \quad \mathbb{E}\left[\frac{|P_n^{(k)}(X_1)|}{|P_n'(X_1)| k!} \,\middle|\, X_1 = z \right] \le \binom{n-1}{k-1} \mathbb{E}\left[\frac{r}{|z - X_2|} \right]^{k-1}$$

Disk Model

Proof of Upper Bound

$$\mathbb{E}[C(\Lambda_n)] \leq C_2\sqrt{n}$$



Area of $(\mathbb{D} - \Lambda_n)$
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
0
1
$r_n = 1 - \sqrt{\kappa \log n / n}$

Components and Critical Points

Lemma: Let $P(z)$ be a monic polynomial and $\Lambda_P(R)$ its R lemniscate. Let $\{\beta_j\}_{j=1}^{n-1}$ denote the critical points of P.

Components and Critical Points

Lemma: Let $P(z)$ be a monic polynomial and $\Lambda_P(R)$ its R lemniscate. Let $\{\beta_j\}_{j=1}^{n-1}$ denote the critical points of P.

$$C(\Lambda_P(R)) = 1 + \#\textbf{Critical points outside } \Lambda_P(R)$$

no. of components

$$= 1 + \left|\left\{ j : |P(\beta_j)| \geq R \right\}\right|$$

- **Question:** Why do critical points contribute to components?

Example:

Polynomial: $P(z) = z^2(z^3 - 1)(z^2 + 2) = z^7 + 2z^5 - z^4 - 2z^2.$

Example:

Polynomial: $P(z) = z^2(z^3 - 1)(z^2 + 2) = z^7 + 2z^5 - z^4 - 2z^2.$

$$P'(z) = 7z^6 + 10z^4 - 4z^3 - 4z.$$

Example:

Polynomial: $P(z) = z^2(z^3 - 1)(z^2 + 2) = z^7 + 2z^5 - z^4 - 2z^2.$

$$P'(z) = 7z^6 + 10z^4 - 4z^3 - 4z.$$

Critical point	**Critical value**	**Modulus**
$-0.060312 + 1.185215i$	$0.830736 + 1.334413i$	1.571872
$-0.060312 - 1.185215i$	$0.830736 - 1.334413i$	1.571872
0.766273	-0.835615	0.835615
$-0.322825 + 0.652134i$	$0.403398 + 0.408102i$	0.573827
$-0.322825 - 0.652134i$	$0.403398 - 0.408102i$	0.573827
0	0	0

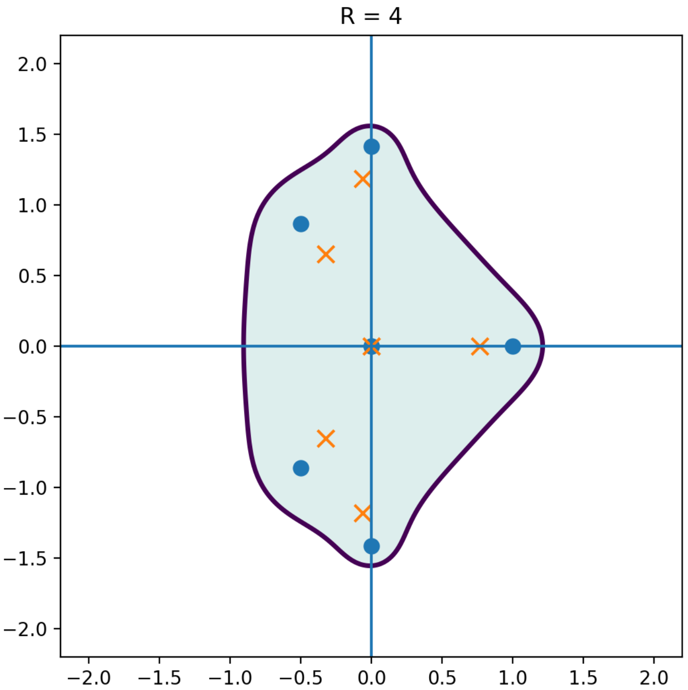
Example:

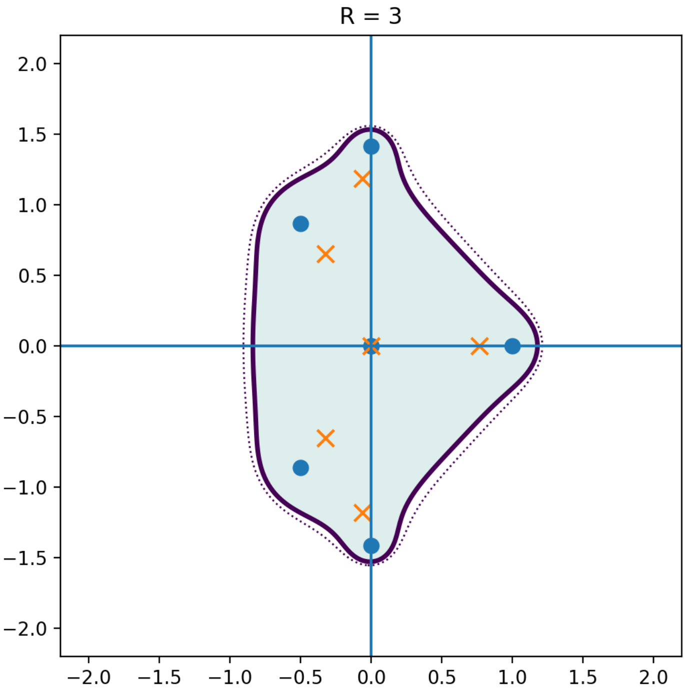
Polynomial: $P(z) = z^2(z^3 - 1)(z^2 + 2) = z^7 + 2z^5 - z^4 - 2z^2$.

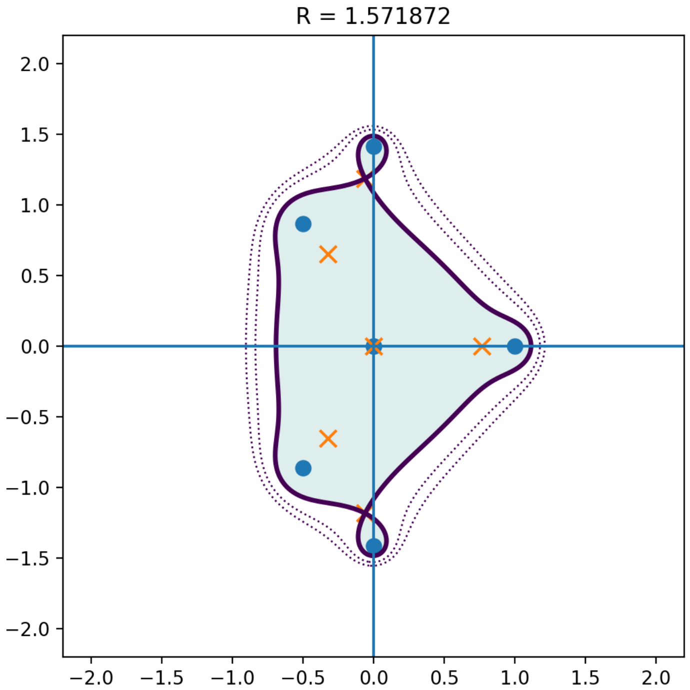
$$P'(z) = 7z^6 + 10z^4 - 4z^3 - 4z.$$

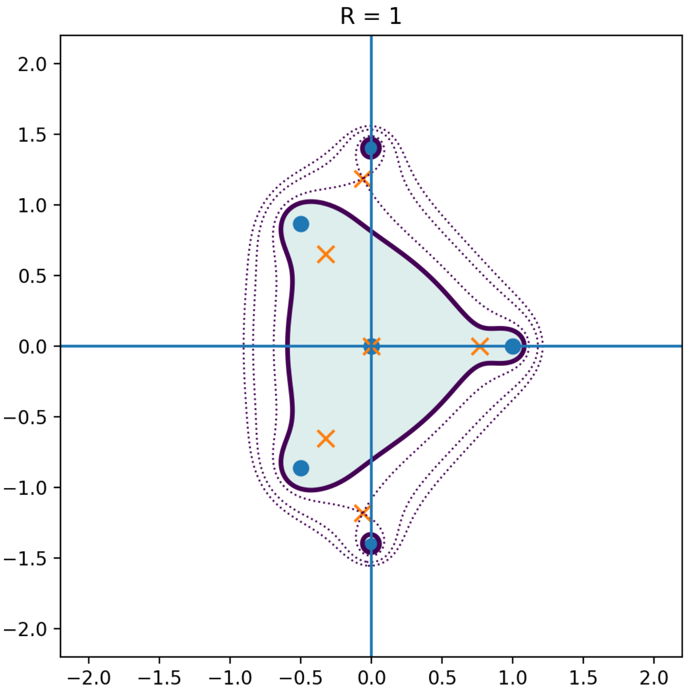
Critical point	**Critical value**	**Modulus**
$-0.060312 + 1.185215i$	$0.830736 + 1.334413i$	1.571872
$-0.060312 - 1.185215i$	$0.830736 - 1.334413i$	1.571872
0.766273	-0.835615	0.835615
$-0.322825 + 0.652134i$	$0.403398 + 0.408102i$	0.573827
$-0.322825 - 0.652134i$	$0.403398 - 0.408102i$	0.573827
0	0	0

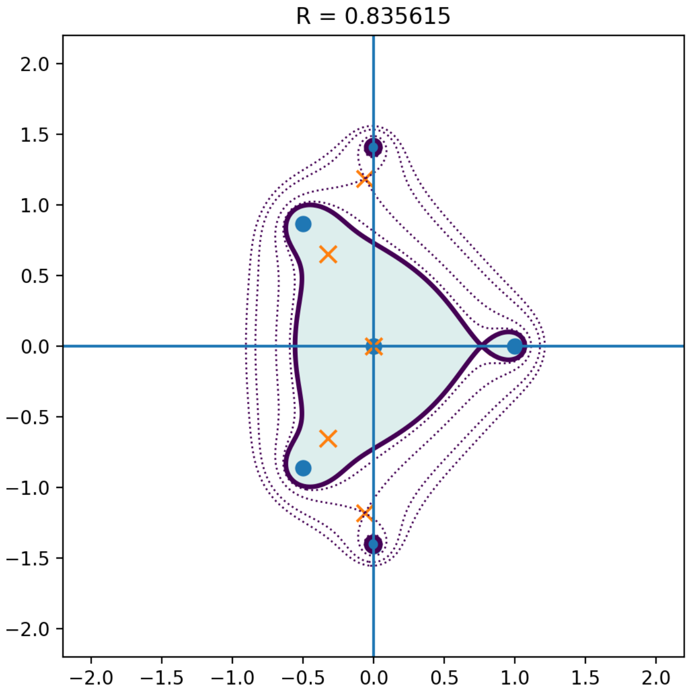
- In this example the unit lemniscate has 3 components!

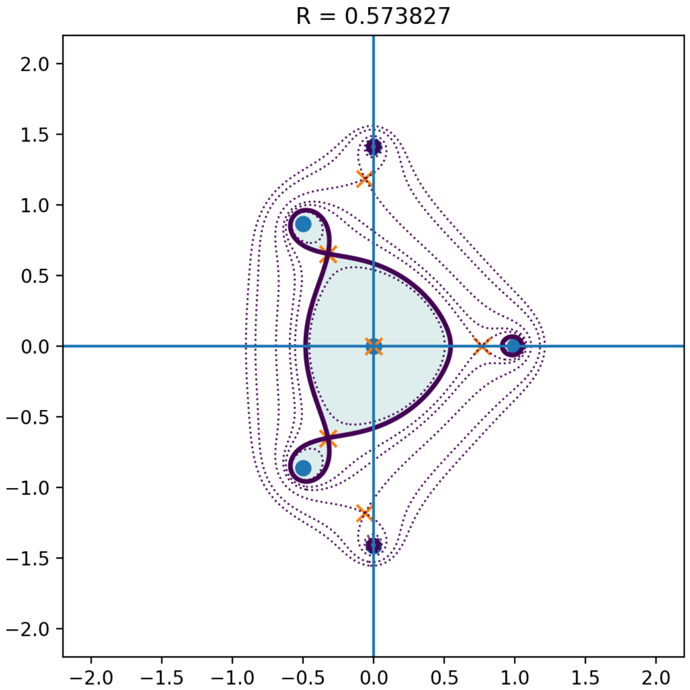

R = 4
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

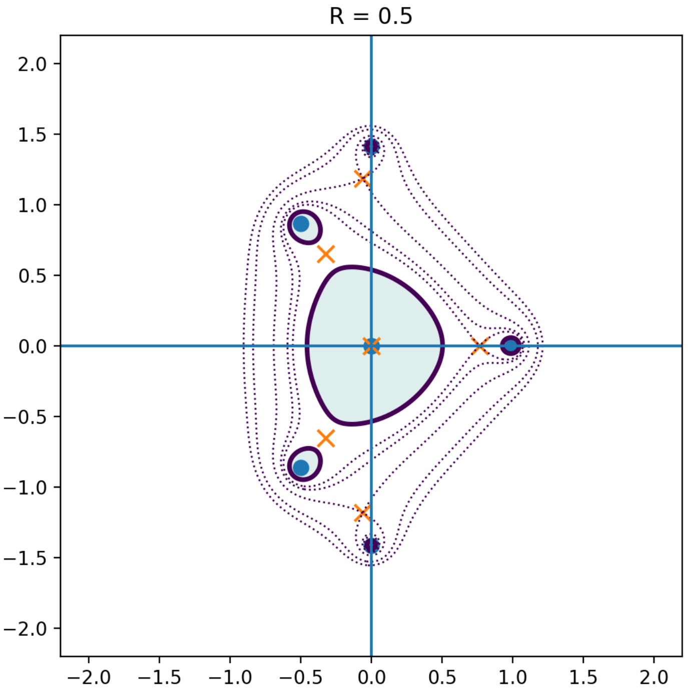

R = 3
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

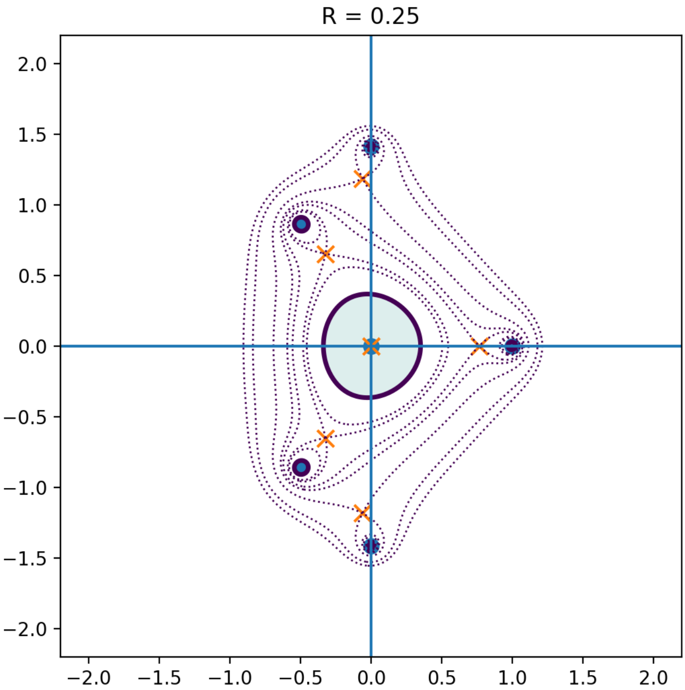

R = 1.571872
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0


R = 1
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0


R = 0.835615
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0


R = 0.573827
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0


R = 0.5
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0 −1.5 −1.0 −0.5 0.0 0.5 1.0 1.5 2.0


R = 0.25
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

Components and Critical Points

Let $P_n(z)$ be a random polynomial, and let $\{\beta_j\}_{j=1}^{n-1}$ denote the critical points of P_n.

Components and Critical Points

Let $P_n(z)$ be a random polynomial, and let $\{\beta_j\}_{j=1}^{n-1}$ denote the critical points of P_n.

$$C(\Lambda_{P_n}) = 1 + \#\{\textbf{critical points outside } \Lambda_{P_n}\}$$

no. of components

$$= n - \#\{\textbf{critical points inside } \Lambda_{P_n}\}.$$

Components and Critical Points

Let $P_n(z)$ be a random polynomial, and let $\{\beta_j\}_{j=1}^{n-1}$ denote the critical points of P_n.

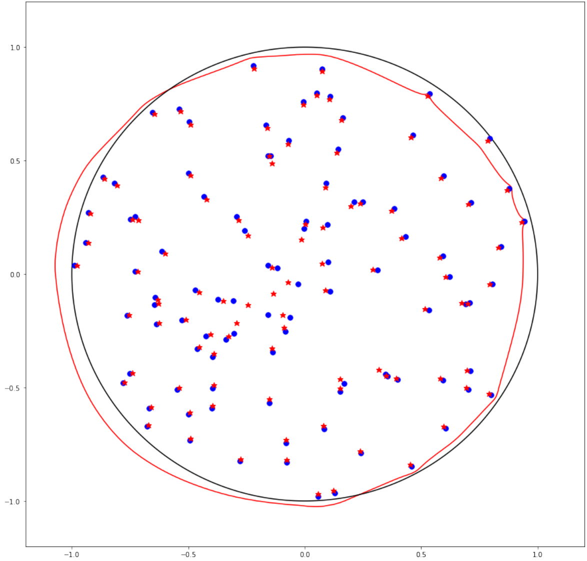
$$C(\Lambda_{P_n}) = 1 + \#\{\textbf{critical points outside } \Lambda_{P_n}\}$$

no. of components

$$= n - \#\{\textbf{critical points inside } \Lambda_{P_n}\}.$$

- **Question:** How do we compute the critical points and critical values of the random polynomial P_n?

Pairing of zeroes and critical points in Random polynomials:



Pairing Continued

Theorem (Kabluchko–Seidel, EJP, 2019)

Assume that μ has a Lebesgue density which is continuous on some open set. Suppose also that its Cauchy transform is non-zero a.e.

Pairing Continued

Theorem (Kabluchko–Seidel, EJP, 2019)

Assume that μ has a Lebesgue density which is continuous on some open set. Suppose also that its Cauchy transform is non-zero a.e. Let $r_n > 0$ satisfy

$$nr_n \longrightarrow +\infty \qquad \text{and} \qquad \sqrt{n}\, r_n \longrightarrow 0 .$$

scale of pairing

Pairing Continued

Theorem (Kabluchko–Seidel, EJP, 2019)

Assume that μ has a Lebesgue density which is continuous on some open set. Suppose also that its Cauchy transform is non-zero a.e. Let $r_n > 0$ satisfy

$$nr_n \longrightarrow +\infty \qquad \text{and} \qquad \sqrt{n}\, r_n \longrightarrow 0 .$$

scale of pairing

Then

$$\lim_{n\to\infty} \mathbb{P}\left[\begin{array}{c} \text{there is a unique critical point} \\ \text{in } B(X_1, r_n) \end{array} \right] = 1$$

Good Root

- We say that a root X_1 is *good* if there exists $r > 0$ (for us $r = n^{-3/4}$ is good enough) such that

$$\begin{cases} B(X_1, r) \subset \Lambda_n \\ \text{ball lies inside the lemniscate} \\ \min_{2 \le j \le n} |X_1 - X_j| > 3r \\ \text{no other roots nearby} \\ \exists \text{ a unique critical point } \xi \in B(X_1, r). \\ \text{paired critical point inside ball} \end{cases}$$

Good Root

- We say that a root X_1 is *good* if there exists $r > 0$ (for us $r = n^{-3/4}$ is good enough) such that

$$\begin{cases} B(X_1, r) \subset \Lambda_n \\ \text{ball lies inside the lemniscate} \\ \min_{2 \le j \le n} |X_1 - X_j| > 3r \\ \text{no other roots nearby} \\ \exists \text{ a unique critical point } \xi \in B(X_1, r). \\ \text{paired critical point inside ball} \end{cases}$$

- Let G_j be the event that X_j is a good root. With this definition we immediately get

$$C(\Lambda_n) \le n - \sum_{j=1}^{n} \mathbb{1}_{G_j}.$$

Theorem 3 (S. Ghosh, A. Shekhar and K. Ramachandran, 2026+)

Let $\mu = \mathrm{unif}(\mathbb{D})$. Then

$$\frac{\mathbb{E}[C(\Lambda_n)]}{\sqrt{n}} = \gamma + O\left(\frac{1}{\log n}\right).$$

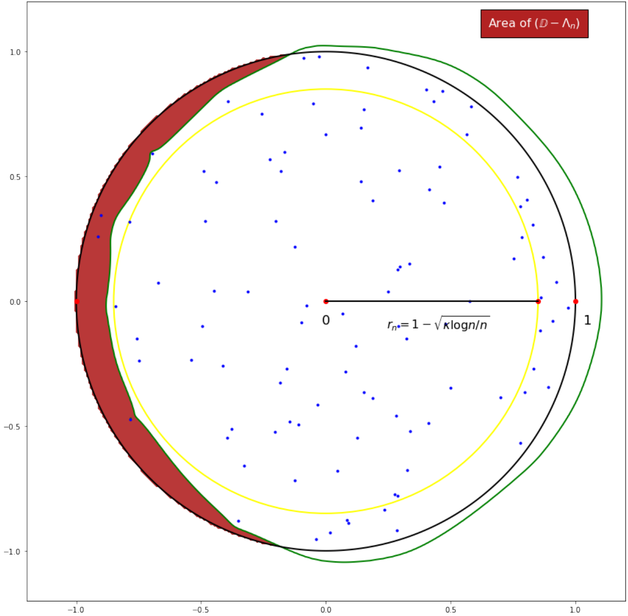
In particular,

$$\lim_{n\to\infty} \frac{\mathbb{E}[C(\Lambda_n)]}{\sqrt{n}} = \gamma.$$

where,

$$\gamma = \sqrt{\frac{\mathrm{Var}(\log|1 - X_1|)}{2\pi}} = \sqrt{\frac{\zeta(2) - 1}{\pi}}.$$

Area Outside The Lemniscate



References

- P. ERDÖS, F. HERZOG, AND G. PIRANIAN, Metric properties of polynomials, J. Analyse Math., 6 (1958), pp. 125–148.
- Z. KABLUCHKO AND H. SEIDEL, Distances between zeroes and critical points for random polynomials with i.i.d. zeroes, Electron. J. Probab., 24 (2019), pp. Paper No. 34, 25.
- M. KRISHNAPUR, E. LUNDBERG, AND K. RAMACHANDRAN, Inradius of random lemniscates, Journal of Approximation Theory (2024).
- E. LUNDBERG AND K. RAMACHANDRAN, The arc length and topology of a random lemniscate, J. Lond. Math. Soc. (2), 96 (2017), pp. 621–641.

Thank you for your attention!!